Commutative Algebra
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1 Ext Continued

Proposition: Let P be an R-module. P is projective iff $Ext^n_R(P, N) = 0$

Proof:
[$\Rightarrow$] $\to 0 \to 0 \to P \to P \to P \to 0$
is a projective resolution of P. Apply Hom(-,N) to the truncated resolution to get
$0 \to Hom(P, N) \to 0 \to 0 \to ..$
so for $n \geq 1, Ext^n_R(P, N) = 0$.
[$\Leftarrow$] Let $\mathcal{P}$ be a truncated projective resolution of P. Take
$0 \to A \to B \to C \to 0$
a short exact sequence of R-modules. Then there is a sequence of chain complexes:
$0 \to Hom(\mathcal{P}, A) \to Hom(\mathcal{P}, B) \to Hom(\mathcal{P}, C) \to 0$
which is exact since each P_i in $\mathcal{P}$ is projective. So we have a long exact sequence in homology, namely
$0 \to Ext^0(P, A) \to Ext^0(P, B) \to Ext^0(P, C) \to Ext^1(P, A) \to ...$
and
$Ext^0(P, A) = Hom(P, A)$, $Ext^0(P, B) = Hom(P, B)$, $Ext^0(P, C) = Hom(P, C)$, and $Ext^1(P, A) = 0$.
So $Hom(P, -)$ is exact, thus P is projective.

This proposition along with the other 2 from last time characterize Ext.

Proposition: $Ext^0(M, N) \cong Hom(M, N)$

Proposition: Suppose $0 \to A \to B \to C \to 0$ is exact. Let N be another R-module.
Then $0 \to Ext^0(C, N) \to Ext^0(B, N) \to Ext^0(A, N) \to Ext^1(C, N) \to ...$ is exact.

$\underline{\text{Theorem:}}$ Let $\underline{Ext^n} : R-mod \to Ab$ be a sequence of contravariant functors such that:
1. $\forall 0 \to A \to B \to C \to 0$ short exact $\exists$
$\to \underline{Ext^n(C)} \to \underline{Ext^n(B)} \to \underline{Ext^n(A)} \xrightarrow{\Delta_n} \underline{Ext^{n+1}(C)} \to ...$
exact with Δ_n natural.
2. $\exists$ R-module N such that $\underline{Ext^0}$ and $Hom(-, N)$ are naturally equivalent.
3. $\underline{Ext^n}(P) = 0 \; \forall \; P$ projective, $\forall n \geq 1$.
Then if E^n is another sequence of contravariant functors satisfying the same axioms with the same N in 2. then $\underline{Ext^n}$ and E^n are naturally isomorphic.

Corollary: Ext as we defined it previously (note, we haven't proved it, because we skipped naturality in the propositions) is independent of the choice of projective resolution.

Proof of Theorem: Naturality will not be checked.
By induction on n.
Base Case: True by 2.
Given a module A, build an exact sequence
$0 \to L \to P \to A \to 0$ with P projective.
By 1. the following rows are exact:

$$\begin{array}{ccccccc} \underline{Ext^0}(P) & \to & \underline{Ext^0}(L) & \xrightarrow{\Delta_0} & \underline{Ext^1}(A) & \to & \underline{Ext^1}(P) \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom(P,N) & \to & Hom(L,N) & \xrightarrow{\delta_0} & E(A) & \to & E^1(A) \end{array}$$

Where the down arrows indicate:
$\underline{Ext^0}(P) \xrightarrow{\cong} Hom(P, N), Ext^n n(L) \xrightarrow{\cong} Hom(L, N)$. The isomorphisms are from 2 and the diagram commutes by naturality. By 3. $Ext^1(P) = 0, E^1(P) = 0$. Thus by the 5-lemma we get an isomorphism σ from $\underline{Ext^1}(A)$ to $E(A)$.
Now, take $n \geq 1$.

$$\begin{array}{ccccccc} 0 = \underline{Ext^n}(P) & \to & \underline{Ext^n}(L) & \xrightarrow{\Delta_n} & \underline{Ext^{n+1}}(A) & \to & \underline{Ext^{n+1}} = 0 \\ & & \downarrow & & & & \\ 0 = E^n(P) & \to & E^n(L) & \to & E^{n+1}(A) & \to & E^{n+1}(P) = 0 \end{array}$$

Here, the downwards arrow indicates $Ext^n(L) \cong E^n(L)$
By induction we get $\underline{Ext^n}(L) \cong E^n(L)$.
By exactness of rows Δ_n, δ_n are also isomorphisms, giving σ an isomorphism.
This last trick comes up frequently. It is called dimension shifting.

Proposition: Suppose $0 \to A \to P \to C \to 0$ is a short exact sequence with P projective. Then $\forall$ modules N, $\forall n \geq 1, Ext^{n+1}(C, N) \cong Ext^n(A, N)$.

Proof: From the long exact sequence
$0 = Ext^n(P, N) \to Ext^n(A, N) \to Ext^{n+1}(C, N) \to Ext^{n+1}(P, N) = 0$
So $Ext^n(A, N) \cong Ext^{n+1}(C, N)$.
Here's an example of something you can do by dimension shifting.

Proposition: Let M, N be R-modules.
Let $\to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be a projective resolution of M.
Let $K_i = ker(d_i)$.
Then there is an exact sequence
$0 \to Hom(K_{n-2}, N) \to Hom(P_{n-1}, N) \to Hom(K_{n-1}, N) \to Ext^n(M, N) \to 0$.

Proof: $0 \to K_{n-1} \to P_{n-1} \to K_{n-2} \to 0$ is a short exact sequence, since the projective resolution is exact. So we get
$0 \to Hom(K_{n-2}, N) \to Hom(P_{n-1}, N) \to Hom(K_{n-1}, N) \to Ext^1(K_{n-2}, N) \to Ext^1(P_{n-1}, N) = 0$.
By dimension shifting,
$Ext^1(K_{n-2}, N) \cong Ext^2(K_{n-3}, N) \cong Ext^3(K_{n-4}, N)... \cong Ext^{n-1}(K_0, N) \cong Ext^n(M, N)$.

2 Ext and Direct Sums

Note: Rotman writes Σ for $\oplus$.

Proposition: Let $\{M_i : i \in I\}$ be a family of modules and N be a module. Then $\forall n$, $Ext^n(\oplus M_i, N) \cong \prod Ext^n(M_i, N)$.

Proof: n=0: $Ext^0(\oplus M_i, N) = Hom(\oplus M_i, N) \prod Ext^0(\oplus M_i, N) = \prod Hom(M_i, N)$.
These are isomorphic as follows:
Take $(f_i)_{i \in I} \in \prod Hom(M_i, N)$, $f_i \in Hom(M_i, N)$.
View $(f_i) : \oplus M_i \to N$ via $(f_i)(\Sigma m_j) = \Sigma f_j(m_j) \in N$, where the sums are finite.
If $f \in Hom(\oplus M_i, N)$,
then $(f|_{M_i})_{i \in I} \in \prod Hom(M_i, N)$.
That is the base case.
For each $i \in I$, take
$0 \to L_i \to P_i \to M_i \to 0$ short, exact, with P_i projective.
Then
$0 \to L_i \to P_i \to M_i \to 0$ is also short, exact, and $\oplus P_i$ is projective.
Then
$Hom(\oplus P_i, N) \to Hom(\oplus L_i, N) \to Ext^1(\oplus M_i, N) \to 0$
$\downarrow \qquad\qquad\qquad \downarrow$
$\prod Hom(P_i, N) \to \prod Hom(L, N) \to \prod Ext^1(M_i, N) \to 0$.
Here the downwards arrows indicate isomorphisms.
So by the 5-lemma: $Ext^1(\oplus M_i, N) \prod Ext^1(M_i, N)$.
In general, by induction and dimension shifting, we get the result.

Proposition: Let $\{N_i : i \in I\}$ be a family of modules, and M another module. Then $Ext^n(M, \prod N_i) \cong \prod Ext^n(M, N_i)$.

Proof: Ommited. Essentially dual to previous, but needs injective modules in place of projective ones.

Corollary: Ext commutes with finite direct sums in either varible.

Proof: $\prod$ is equivalent to $\oplus$ in the finite case.

3 What does Ext^1 look like?

Proposition: Let G be an Abelian Group. Then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

Proof: From $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$.
We get a long exact sequence:

$$\begin{array}{ccccccc} Hom(\mathbb{Z}, G) & \to & Hom(\mathbb{Z}, G) & \to & Ext^1(\mathbb{Z}/n\mathbb{Z}) & \to & Ext^1(\mathbb{Z}, G) = 0 \\ \downarrow & & \downarrow & & & & \\ G & \to & G & \to & G/nG & \to & 0 \end{array}$$

where the downwards arrows represent $Hom(\mathbb{Z}, G) \xrightarrow{\cong}, G$.
By 5-lemma, we get $Ext^1(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

Definition: Let C and A be R-modules. An extension of A by C is a short exact sequence
$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$.
The extension is split if the sequence is split.

Idea: B is the extension $A \cong i(A) \subseteq B$. So A is in B, but B is bigger by C.

Proposition: If $Ext^1(C, A) = 0$ then every extension of A by C splits.

Proof: Suppose we have an extension
$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$.
Then
$Hom(C, B) \xrightarrow{p_*} Hom(C, C) \to Ext^1(C, A) = 0$.
So p_* is surjective, so $\exists s \in Hom(C, B)$ with $ps = p_* s = 1_c$. But this is the splitting map.
The converse is also true, but we'll do that another time.

Corollary: An R-module P is projective iff $\forall B$ R-module, $Ext^1(P, B) = 0$.

Proof:
[$\Rightarrow$] We already know.
[$\Leftarrow$] Given an exact sequence, $0 \to B \to X \to P \to 0$
it splits by the proposition and so P is projective.